Resonantly Induced Friction and Frequency Combs in Driven Nanomechanical Systems

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(Received 20 March 2019; revised manuscript received 10 May 2019; published 27 June 2019)

We propose a new mechanism of friction in resonantly driven vibrational systems. The form of the friction force follows from the time- and spatial-symmetry arguments. We consider a microscopic mechanism of this resonant force in nanomechanical systems. The friction can be negative, leading to the onset of self-sustained oscillations of the amplitude and phase of forced vibrations, which result in a frequency comb in the power spectrum.

DOI: 10.1103/PhysRevLett.122.254301

The physics of friction keeps attracting attention in diverse fields and at different spatial scales, from cold atoms to electrons on helium to locomotion of devices and animals [1–6]. An important type of systems where friction plays a critical role and which has been studied in depth, both theoretically and experimentally, are vibrational systems. The simplest form of friction in these (and many other) systems is viscous friction. For a vibrational mode with the coordinate q, the viscous friction force is $\propto \dot{q}$. It describes a large number of experiments on various kinds of vibrational systems, nano- and micromechanical modes and electromagnetic cavity modes being examples of the particular recent interest [7,8].

In vibrational systems, viscous friction is often called linear friction, to distinguish it from nonlinear friction, which nonlinearly depends on q and $\dot{q}$. Phenomenologically, the simplest nonlinear friction force is $\propto q^2\dot{q}$ (the van der Pol form [9]) or $\propto\dot{q}^3$ (the Rayleigh form [10]). Both these forms of the force are particularly important for weakly damped systems. This is because in such systems the vibrations are nearly sinusoidal, whereas both forces have resonant components which oscillate at the mode frequency. Moreover, both forces lead to the same long-term dynamics of a weakly damped mode and in this sense are indistinguishable [11,12].

External driving of vibrational modes can modify their dissipation. The change has been well understood for a periodic driving tuned sufficiently far away from the mode eigenfrequency. Such driving can open new decay channels where transitions between the energy levels of the mode are accompanied by absorption or emission of excitations of the thermal reservoir and a drive quantum $\hbar\omega_F$, with ω_F being the drive frequency [13]. This can lead to both linear [14,15] and nonlinear friction [16,17]. It has been also found that, in microwave cavities and nanomechanical systems, resonant driving can reduce linear friction by slowing down energy transfer from the vibrational mode to two-level systems due to their saturation [18–20].

In this Letter, we consider nonlinear friction induced by resonant driving, which significantly differs from other forms of friction. We show that, in nanomechanical systems, the proposed friction can become important already for a moderately strong drive and can radically modify the response to the drive, including the onset of slow oscillations of the amplitude and phase of the driven mode with the increasing drive.

Phenomenologically, a mode with inversion symmetry driven by a force $F(t) = F\cos\omega_F t$ can experience a resonant induced friction force (RIFF) of the form

$$f_{\mathrm{RIFF}} = -\eta_{\mathrm{RIFF}} F(t) q\dot{q}. \tag{1}$$

Such force has the proper spatial symmetry, as it changes sign on spatial inversion ($q \to -q$ and $F \to -F$) and is dissipative, as it changes sign on time inversion $t \to -t$. The driving frequency ω_F is assumed to be close to the mode eigenfrequency ω_0, so that the force f_{RIFF} has a resonant component, as $F(t)$, $q(t)$, and $\dot{q}(t)$ all oscillate at frequencies equal or close to ω_F. The friction coefficient η_{RIFF} is undetermined in the phenomenological theory. It can be positive or negative, as the very onset of the force f_{RIFF} is a nonequilibrium phenomenon. Therefore, f_{RIFF} can either increase or decrease the decay rate, or even make it negative, in a certain parameter range.

The form of the RIFF reminds the form of the van der Pol friction force, except that q^2 is replaced by $F(t)q$. In some sense, the force $F(t)$ is "smaller" than the displacement q near resonance: this is the well-known effect that a small resonant force leads to large vibration amplitude for weak damping. Therefore, f_{RIFF} can be significant if there is a mechanism that compensates the relative smallness of $F(t)$.

For nanomechanical resonators, a simple microscopic mechanism of the RIFF is heating. The absorbed power $F(t)\dot{q}$ leads to a temperature change δT, which can be relatively large due to the small thermal capacity of a nanoresonator (generally, the temperature change depends on the coordinates in the resonator [21]). In turn, the temperature change modifies the resonator eigenfrequency ω_0, e.g., due to thermal expansion, cf. [26,27]. To the lowest order in δT, the eigenfrequency change is $\delta\omega_0 = -\lambda_\omega \delta T$. The coefficient λ_ω depends on the material and the spatial structures of the mode and the temperature field.

In many cases, the relaxation time of the temperature in the resonator is much longer then the vibration period $t_F = 2\pi/\omega_F$. Then the temperature change is proportional to the period-averaged power

$$\delta T(t) = \lambda_T [F(t)\dot{q}(t)]_{\mathrm{av}} \equiv \lambda_T t_F^{-1} \int_t^{t+t_F} dt' F(t')\dot{q}(t'),$$

(in fact, δT is spatially nonuniform [21]). As a result, the restoring force $-m\omega_0^2 q$ is incremented by f_T,

$$f_T(t) = 2m\omega_0 \lambda_\omega \lambda_T [F(t)\dot{q}(t)]_{\mathrm{av}} q(t). \tag{2}$$

The force $f_T(t)$ is a specific form of the RIFF. The thermal mechanism is not the only RIFF mechanism, but it is often important, and moreover, the ratio of the conventional nonlinear friction to the RIFF contains a small parameter [21].

We now consider the dynamics of a driven nanoresonator in the presence of RIFF. Nanoresonators are often well described by the Duffing model, which takes into account quartic nonlinearity [11], but the analysis below immediately extends to other nonlinearity mechanisms, cf. [28]. The Hamiltonian of the Duffing oscillator in the absence of coupling to the thermal reservoir is

$$H_0 = \frac{1}{2}(p^2 + \omega_0^2 q^2) + \frac{1}{4}\gamma q^4 - qF\cos\omega_F t. \tag{3}$$

Here p is the oscillator momentum. We scaled the variables so that the mass is $m = 1$. For concreteness, we assume that the Duffing nonlinearity parameter γ is positive. The driving is assumed resonant, $|\omega_F - \omega_0| \ll \omega_0$, and comparatively weak, so that $|\gamma|\langle q^2\rangle \ll \omega_0^2$.

To analyze the behavior on the timescale long compared to ω_F^{-1}, one can change to the rotating frame and introduce slowly varying in time canonically conjugate coordinate q_0 and momentum p_0 (the analogs of the quadrature operators [7])

$$q(t) + i\omega_F^{-1} p(t) = (\omega_F)^{-1/2}(q_0 + ip_0)\exp(-i\omega_F t).$$

In the standard rotating wave approximation (RWA), from Eq. (3) we obtain Hamiltonian equations for q_0, p_0 with the time-independent Hamiltonian H_{RWA}

$$(\dot{q}_0)_H = \partial_{p_0} H_{\mathrm{RWA}}, \qquad (\dot{p}_0)_H = -\partial_{q_0} H_{\mathrm{RWA}},$$
$$H_{\mathrm{RWA}}(q_0, p_0) = -\frac{1}{2}\delta\omega(q_0^2 + p_0^2) + \frac{3\gamma}{32\omega_F^2}(q_0^2 + p_0^2)^2 - Fq_0/2\sqrt{\omega_F}, \qquad \delta\omega = \omega_F - \omega_0. \tag{4}$$

The value of H_{RWA} gives the quasienergy of the driven nanoresonator in the RWA.

It is well known how to incorporate linear friction into the RWA equations of motion starting from both a microscopic formulation and the phenomenological friction force $-2\Gamma\dot{q}$ [29–32]. An extension to the RIFF is straightforward. Keeping only smoothly varying terms in the equations for $\dot{q}_0, \dot{p}_0$, in the case of the heating-induced RIFF (2) we obtain the following equations of motion

$$\dot{q}_0 = -\Gamma q_0 - J_T p_0^2 + \partial_{p_0} H_{\mathrm{RWA}},$$
$$\dot{p}_0 = -\Gamma p_0 + J_T q_0 p_0 - \partial_{q_0} H_{\mathrm{RWA}}. \tag{5}$$

Here $J_T = \omega_F^{1/2} F \lambda_\omega \lambda_T / 2$. In Eq. (5), we have disregarded noise. It is typically weak in weakly damped nanoresonators and leads primarily to small fluctuations about the stable states of forced vibrations and occasional switching between the stable states in the range of bistability, cf. [32–38] and references therein; here we do not consider these effects.

Parameter J_T that characterizes the RIFF increases with the driving amplitude F; the RIFF also increases with the vibration amplitude $A = [(q_0^2 + p_0^2)/\omega_F]^{1/2}$. From Eq. (5), the effects of the RIFF become pronounced for $|J_T A|\omega_F^{1/2} \sim \Gamma$ and should be seen already for a moderately strong drive if the decay rate Γ due to the linear friction is small.

If both the linear friction and the RIFF can be disregarded, the values $(q_{\mathrm{st}}, p_{\mathrm{st}})$ of (q_0, p_0) at the stationary states of forced vibrations are given by the conditions $\partial_{q_0} H_{\mathrm{RWA}} = \partial_{p_0} H_{\mathrm{RWA}} = 0$, which reduce to equations

$$\frac{3\gamma}{8\omega_F^2} q_{\mathrm{st}}^3 - \delta\omega q_{\mathrm{st}} = F/2\sqrt{\omega_F}, \qquad p_{\mathrm{st}} = 0. \tag{6}$$

The equation for q_{st} has one real root in the range of $F, \delta\omega$ where the oscillator is monostable in the weak dissipation limit or three real roots in the range of bistability. In the latter range, of primary interest for the analysis of the RIFF is the root with the maximal q_{st}, and in what follows q_{st} refers to this root. For small Γ and $J_T = 0$ it corresponds to a stable state of forced vibrations at frequency ω_F, as does also the real root q_{st} in the range of monostability [39]. In the both cases, the considered $(q_{\mathrm{st}}, p_{\mathrm{st}})$ corresponds to the minimum of H_{RWA}.

For $J_T > 0$, the RIFF can lead to an instability of the forced vibrations. Indeed, to the leading order in Γ, J_T, the sum of the eigenvalues of Eq. (5) linearized about the stable

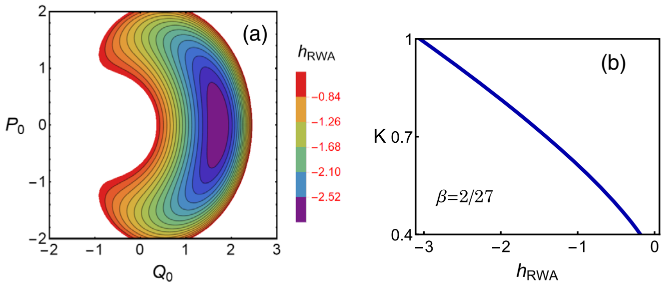

FIG. 1. (a) The Hamiltonian trajectories (4) for different values of the scaled RWA energy $h_{\rm RWA} = (6\gamma/F^4)^{1/3} H_{\rm RWA}$, Eq. (10). The scaled field strength defined in Eq. (11) is $\beta = 2/27$. The driven oscillator is bistable for this β and shown are the trajectories that circle the large-amplitude state at the minimum of $h_{\rm RWA}$ ($Q_0 \approx 1.72, P_0 = 0$). This state becomes stable in the presence of weak linear friction. For other values of β, the trajectories not too close to the minimum of $h_{\rm RWA}$ also have a horse-shoe form. (b) The scaled ratio of the decay and gain rates K, Eq. (9), as a function of $h_{\rm RWA}$.

state is $-2\Gamma + J_T q_{\rm st}$. When this sum becomes equal to zero, the system undergoes a supercritical Hopf bifurcation. This means that, for $J_T q_{\rm st} > 2\Gamma$, the state of forced vibrations with constant amplitude and phase becomes unstable. The amplitude and phase oscillate in time, which correspond to oscillations of the system in the rotating frame about $(q_{\rm st}, p_{\rm st})$.

For small Γ and $J_T q_{\rm st}$ (the condition is specified below), one can think of the steady motion in the rotating frame as occurring with a constant value of the Hamiltonian $H_{\rm RWA}$ along the Hamiltonian trajectory (4); see Fig. 1(a). This value is determined by the balance of the damping $\propto \Gamma$ and the RIFF. The dissipative losses $\propto \Gamma$ drive $H_{\rm RWA}$ toward its minimum, whereas the RIFF pumping increases $H_{\rm RWA}$. The stable value of $H_{\rm RWA}$ can be found by averaging over the trajectories (4) the equation of motion for $H_{\rm RWA}(q_0, p_0)$, which follows from Eq. (5). We denote such averaging by an overline

$$\overline{U(t)} = \frac{1}{\mathbb{T}(H_{\rm RWA})} \int_t^{t+\mathbb{T}(H_{\rm RWA})} dt' U(t'; H_{\rm RWA}),$$

where $U(t; H_{\rm RWA})$ is a function calculated along the trajectory (4) for a given value of $H_{\rm RWA}$, and $\mathbb{T}(H_{\rm RWA})$ is the period of motion along this trajectory. After straightforward algebra, we obtain from Eq. (5)

$$\overline{dH_{\rm RWA}/dt} = \frac{1}{\mathbb{T}(H_{\rm RWA})} \int_{\mathcal{S}(H_{\rm RWA})} dq_0 dp_0 (-2\Gamma + J_T q_0). \tag{7}$$

Here, $\mathcal{S}(H_{\rm RWA})$ is the area inside the Hamiltonian trajectory (4) with a given $H_{\rm RWA}$.

From Eq. (7), the condition of the balance of gain and loss that gives the stable value of $H_{\rm RWA}$ is

$$(J_T q_{\rm st}/2\Gamma) K = 1, \tag{8}$$

where

$$K = q_{\rm st}^{-1} \int_{\mathcal{S}(H_{\rm RWA})} q_0 dq_0 dp_0 \left(\int_{\mathcal{S}(H_{\rm RWA})} dq_0 dp_0 \right)^{-1}. \tag{9}$$

Parameter K is the ratio of the rates of decay due to the linear friction and gain due to the RIFF. The dependence of K on $H_{\rm RWA}$ is illustrated in Fig. 1(b). Figure 1 is plotted in the scaled variables Q_0, P_0 and for the scaled Hamiltonian $h_{\rm RWA} = (6\gamma/F^4)^{1/3} H_{\rm RWA}$,

$$h_{\rm RWA} = \frac{1}{4}(Q_0^2 + P_0^2)^2 - \frac{1}{2}\beta^{-1/3}(Q_0^2 + P_0^2) - Q_0,$$
$$Q_0 = q_0/\zeta, \quad P_0 = p_0/\zeta, \quad \zeta = (4F/3\gamma)^{1/3}\omega_F^{1/2}. \tag{10}$$

Function $h_{\rm RWA}$ depends only on one dimensionless parameter, the scaled strength of the driving field

$$\beta = 3\gamma F^2/32\omega_F^3(\delta\omega)^3. \tag{11}$$

As seen from Fig. 1(b) and also from Eq. (9), $K = 1$ where $H_{\rm RWA}$ is at its minimum. Importantly, K monotonically decreases with increasing $H_{\rm RWA}$ in a broad range of $H_{\rm RWA}$. This decrease holds both in the range of β where the oscillator is bistable and where it is monostable in the absence of the RIFF. Therefore, in the presence of the RIFF, once the condition of the onset of oscillations in the rotating frame is met, $J_T q_{\rm st} > 2\,\Gamma$, these oscillations are stabilized at the value of $H_{\rm RWA}$ given by $K \equiv K(H_{\rm RWA}) = 2\,\Gamma/J_T q_{\rm st}$. We emphasize that the frequency of these oscillations $2\pi/\mathbb{T}(H_{\rm RWA})$ is small compared to ω_F, yet it exceeds Γ and $J_T q_{\rm st}$.

Parameter $J_T q_{\rm st}$ depends on the amplitude of the driving field F and the frequency ω_F. By varying F and ω_F one can control the stable value of $H_{\rm RWA}$ and thus the amplitude and frequency of the oscillations in the rotating frame. Remarkably, these oscillations become significantly nonsinusoidal already for comparatively small difference between $H_{\rm RWA}$ and its minimal value. This is seen in Fig. 1(a). The profoundly nonelliptical trajectories are a signature of nonsinusoidal vibrations. Formally, the oscillations are described by the Jacobi elliptic functions [40], which allows finding their Fourier components in the explicit form [21].

The instability of the forced vibrations at the drive frequency and the onset of nonlinear self-sustained oscillations in the rotating frame lead to a qualitative change of the power spectrum of the driven oscillator. There emerge multiple equally spaced peaks on the both sides of ω_F that correspond to the vibration overtones in the rotating frame. This frequency comb effect occurs for an isolated mode and is thus qualitatively different from the frequency combs resulting from a linear [41] or nonlinear [42] resonance between vibrational modes in the presence of Duffing nonlinearity.

The spacing between the frequency comb peaks $2\pi/\mathbb{T}(H_{\mathrm{RWA}})$ is small compared to ω_F. The widths of the peaks are determined by phase diffusion due to the noise in a nanoresonator, in particular, thermal fluctuations of H_{RWA} around its stable value and the related fluctuations of the frequency $2\pi/\mathbb{T}(H_{\mathrm{RWA}})$. These fluctuations are efficiently averaged out by the relaxation, the process reminiscent of motional narrowing in nuclear magnetic resonance [32,43]. Therefore, the widths of the peaks should be much smaller than the damping rate Γ [44].

In conclusion, we have shown that, from the symmetry and resonance arguments, a resonantly driven vibrational mode can experience a specific friction force. This force, the RIFF, is nonlinear in the mode coordinate and explicitly depends on the driving force. The RIFF can be negative. In this case, already for a moderately strong resonant drive, it can lead to an instability of forced vibrations of a weakly damped nonlinear mode, qualitatively modifying the response of such a mode to the drive. The instability causes the onset of self-sustained oscillations of the vibration amplitude and phase. In turn, this leads to a frequency comb in the power spectrum of the driven mode. The effect is general and may emerge in various vibrational systems. We have shown that, in nanomechanics, an important microscopic mechanism of the RIFF is associated with the driving-induced spatially nonuniform heating of a nanoresonator and the resulting change of the mode eigenfrequency.

M. I. D. and M. L. R. acknowledge partial support from the NSF (Grant No. DMR-1806473). M. I. D. was also supported in part by the Moore Distinguished Scholar program at Caltech. G. R. and E. M. W. acknowledge partial support from Deutsche Forschungsgemeinschaft through the collaborative research center SFB 767.

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