Supplementary Information

Nonlinear Mode-Coupling in Nanomechanical Systems

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Derivation of the analytical form of the nonlinear coupling coefficients

The flexural motion of a doubly clamped beam generates a uniform strain along the beam axis, because the ends are clamped. This strain gives rise to a uniform tensile stress via Hooke's law, which increases the beam stiffness. This can affect both the frequency of the mode at which the beam is excited (self-tuning) or those of other modes (cross-tuning). Here we derive the relevant equations that describe these processes.

We start by solving for the flexural modes of a doubly clamped beam. The deflection functions of these modes are given by[1]

$$\Phi_{\mathrm{n}}(\xi) = \mathrm{Q}_{\mathrm{n}} \left(\cosh \kappa_n \xi - \cos \kappa_n \xi + \frac{\cosh \kappa_n - \cos \kappa_n}{\sinh \kappa_n - \sin \kappa_n} [\sin \kappa_n \xi - \sinh \kappa_n \xi] \right), \qquad \text{S.1}$$

where ξ is scaled distance along the beam normalized by the beam length, Q_n is the nth mode's amplitude normalization constant ($Q_n \approx 1$), and $\kappa_n = \sqrt[4]{\frac{\rho A}{YI} {\omega_n}^2}$. The Euler-Bernoulli equation in the presence of an axial stress is[2]

$$\frac{\partial^2 u_2}{\partial t^2} + \gamma^2 \frac{\partial^4 u_2}{\partial \xi^4} - \overbrace{\gamma^2 \frac{A}{I}\left(\frac{Tl^2}{Y} + \frac{1}{2}\int_0^1 \left(\frac{\partial u_2}{\partial \xi}\right)^2 d\xi\right)\frac{\partial^2 u_2}{\partial \xi^2}}^{axial\ stress\ contribution} = 0, \qquad \gamma^2 = \frac{YI}{\mu l^4}, \qquad \text{S.2}$$

where u_2 is the beam instantaneous displacement normal to the beam axis, μ is the linear mass density of the beam, Y is the material Young's modulus, I is the beam's areal moment, l the length of the beam, A the cross-sectional area, and T the intrinsic axial tension (stress) of the material. The middle expression is the contribution due to uniform axial stress in the beam. Within this expression, the first term gives the contribution arising from intrinsic tension, and the second accounts for extension along the beam length due to finite oscillation amplitude.

We then decompose the beam motion into its normal modes

$$u_2(\xi, t) = \sum_n \mathcal{A}_n \Phi_\mathrm{n}(\xi)\zeta_\mathrm{n}(\mathrm{t}). \qquad \text{S.3}$$

Substituting equation S.3 into equation S.2, gives

$$\sum_n \mathcal{A}_n \Phi_n \ddot{\zeta}_n + \gamma^2 \sum_n \mathcal{A}_n \Phi_n^{IV} \zeta_n - \gamma^2 \frac{A}{I} \sum_n \mathcal{A}_n \Phi_n'' \zeta_n \left(\frac{Tl^2}{Y} + \frac{1}{2}\int_0^1 \left(\sum_n \mathcal{A}_n \Phi_n' \zeta_n\right)^2 d\xi\right) = 0, \qquad \text{S.4}$$

For a beam excited at two of its modes k and j, we obtain

$$\ddot{\zeta}_\mathrm{k} + \omega_{\mathrm{k},0}^2 \zeta_\mathrm{k} + \eta_\mathrm{k}\omega_{\mathrm{k},0}^2 \zeta_\mathrm{k}\left[X_{kk}\frac{Tl^2}{Y} + \frac{1}{2}\mathcal{A}_k^2 \zeta_k^2 X_{kk}^2 + \mathcal{A}_j^2 \zeta_j^2 \left(\frac{X_{kk}X_{jj}}{2} + \mathrm{X}_{jk}^2\right)\right] = 0, \qquad \text{S.5}$$

where

$$\eta_k = \frac{A}{I}\frac{1}{\int_0^1 \Phi_\mathrm{k}\Phi_\mathrm{k}^{\mathrm{IV}} d\xi}, \qquad X_{\mathrm{nm}} = \int_0^1 \Phi_n' \Phi_m' d\xi, \omega_{\mathrm{k},0}^2 = \int_0^1 \Phi_\mathrm{k}\Phi_\mathrm{k}^{\mathrm{IV}} d\xi * \gamma^2.$$

The summation convention is not used. The resonant frequency of each mode is modified by the intrinsic tension, T, according to

$$\omega_{k,t}^2 = \omega_{k,0}^2\left(1 + \eta_k X_{kk}\frac{Tl^2}{E}\right), \quad \tau_k = \frac{\eta_k}{1 + \eta_k X_{kk}\frac{Tl^2}{E}}. \qquad \text{S.6}$$

Assuming the beam motion is weakly perturbed by the nonlinearity in Eq. (S.2), we then use the harmonic approximation. Substituting $\zeta(t) \cong \cos\omega_{k,t}t$ into Eq. (S.5) gives the required resonant frequency of mode k in the presence of finite oscillation amplitude of modes k and j,

$$\omega_{k,mod}^2 = \omega_{k,t}^2\left(1 + 2\lambda_{kk}\mathcal{A}_k^2 + 2\lambda_{jk}\mathcal{A}_j^2\right). \qquad \text{S.7}$$

Thus, from equation S.5 the change in resonant frequency of mode k is

$$\frac{\Delta\omega_k}{\omega_k} = \lambda_{jk}\mathcal{A}_{max,j}^2, \qquad \text{S.8}$$

assuming that only mode j is driven to high amplitudes, i.e., the tension developed by the oscillation of mode k is insignificant. The coupling coefficients in Eq. (S.9) are

$$\lambda_{jk} = \left(2 - \delta_{jk}\right)\frac{\tau_k}{8}\left(\frac{X_{kk}X_{jj}}{2} + X_{jk}^2\right). \qquad \text{S.9}$$

These coefficients λ_{jk} form the nonlinear stiffness tensor that relates the change in resonant frequency to the amplitude of resonant motion of modes k and j. The diagonal components are the well-known "Duffing"[3] terms of a single mode oscillation. The off-diagonal components describe the nonlinear coupling between two different modes.

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